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International Baccalaureate
Baccalauréat International
Bachillerato Internacional

MATHEMATICS
HIGHER LEVEL
PAPER 2

Candidate session number

Wednesday 14 May 2014 (morning)

2 hours

Examination code

2	2	1	4	–	7	2	0	6

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions in the boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the ***Mathematics HL and Further Mathematics HL*** *formula booklet* is required for this paper.
- The maximum mark for this examination paper is *[120 marks]*.

15 pages

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, for example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A

*Answer **all** the questions in the boxes provided. Working may be continued below the lines if necessary.*

1. *[Maximum mark: 6]*

(a) (i) Find the sum of all integers, between 10 and 200, which are divisible by 7.

(ii) Express the above sum using sigma notation. *[4]*

An arithmetic sequence has first term 1000 and common difference of -6. The sum of the first n terms of this sequence is negative.

(b) Find the least value of n. *[2]*

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2. *[Maximum mark: 5]*

The weights, in kg, of one-year-old bear cubs are modelled by a normal distribution with mean μ and standard deviation σ.

(a) Given that the upper quartile weight is 21.3 kg and the lower quartile weight is 17.1 kg, calculate the value of μ and the value of σ. *[4]*

A random sample of 100 of these bear cubs is selected.

(b) Find the expected number of bear cubs weighing more than 22 kg. *[1]*

3. *[Maximum mark: 5]*

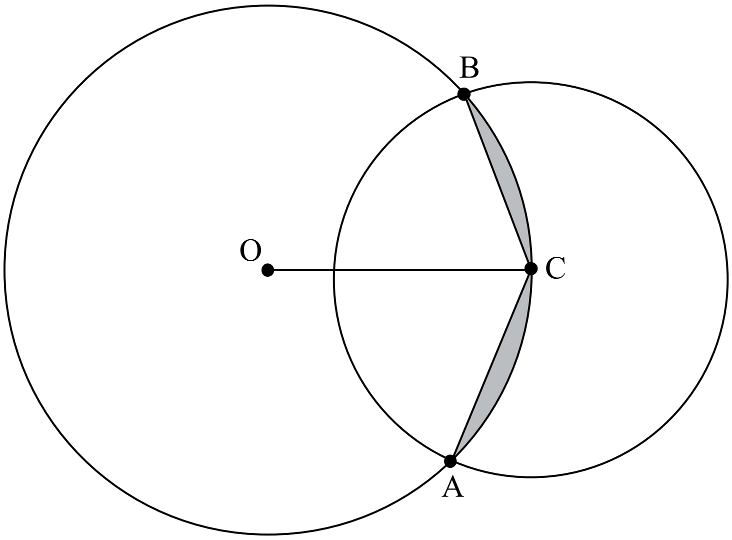
The graphs of $y = x^2 e^{-x}$ and $y = 1 - 2\sin x$ for $2 \le x \le 7$ intersect at points A and B. The x-coordinates of A and B are x_A and x_B.

(a) Find the value of x_A and the value of x_B. *[2]*

(b) Find the area enclosed between the two graphs for $x_A \le x \le x_B$. *[3]*

4. *[Maximum mark: 6]*

The following diagram shows two intersecting circles of radii 4 cm and 3 cm. The centre C of the smaller circle lies on the circumference of the bigger circle. O is the centre of the bigger circle and the two circles intersect at points A and B.



Find:

(a) $B\hat{O}C$; *[2]*

(b) the area of the shaded region. *[4]*

5. *[Maximum mark: 6]*

Find the coefficient of x^{-2} in the expansion of $(x-1)^3\left(\frac{1}{x}+2x\right)^6$.

6. *[Maximum mark: 7]*

Six customers wait in a queue in a supermarket. A customer can choose to pay with cash or a credit card. Assume that whether or not a customer pays with a credit card is independent of any other customers' methods of payment.

It is known that 60 % of customers choose to pay with a credit card.

(a) Find the probability that:

(i) the first three customers pay with a credit card and the next three pay with cash;

(ii) exactly three of the six customers pay with a credit card. *[4]*

There are n customers waiting in another queue in the same supermarket. The probability that at least one customer pays with cash is greater than 0.995.

(b) Find the minimum value of n. *[3]*

7. *[Maximum mark: 8]*

The function f is defined as $f(x) = -3 + \frac{1}{x-2}, x \neq 2$.

(a) (i) Sketch the graph of $y = f(x)$, clearly indicating any asymptotes and axes intercepts.

(ii) Write down the equations of any asymptotes and the coordinates of any axes intercepts. *[4]*

(b) Find the inverse function f^{-1}, stating its domain. *[4]*

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8. *[Maximum mark: 4]*

The random variable X has a Poisson distribution with mean μ.

Given that $\mathrm{P}(X=2)+\mathrm{P}(X=3)=\mathrm{P}(X=5)$,

(a) find the value of μ; *[2]*

(b) find the probability that X lies within one standard deviation of the mean. *[2]*

9. *[Maximum mark: 5]*

Sand is being poured to form a cone of height h cm and base radius r cm. The height remains equal to the base radius at all times. The height of the cone is increasing at a rate of $0.5\ \text{cm min}^{-1}$.

Find the rate at which sand is being poured, in $\text{cm}^3\ \text{min}^{-1}$, when the height is 4 cm.

10. *[Maximum mark: 8]*

Consider the curve with equation $\left(x^2 + y^2\right)^2 = 4xy^2$.

(a) Use implicit differentiation to find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$. *[5]*

(b) Find the equation of the normal to the curve at the point $(1, 1)$. *[3]*

*Do **NOT** write solutions on this page.*

SECTION B

*Answer **all** questions in the answer booklet provided. Please start each question on a new page.*

11. *[Maximum mark: 13]*

The probability density function of a random variable X is defined as:

$$f(x) = \begin{cases} a x \cos x, & 0 \le x \le \dfrac{\pi}{2}, \text{ where } a \in \mathbb{R}. \\ 0, & \text{elsewhere} \end{cases}$$

(a) Show that $a = \dfrac{2}{\pi - 2}$. *[5]*

(b) Find $\mathrm{P}\left(X < \dfrac{\pi}{4}\right)$. *[2]*

(c) Find:

(i) the mode of X;

(ii) the median of X. *[4]*

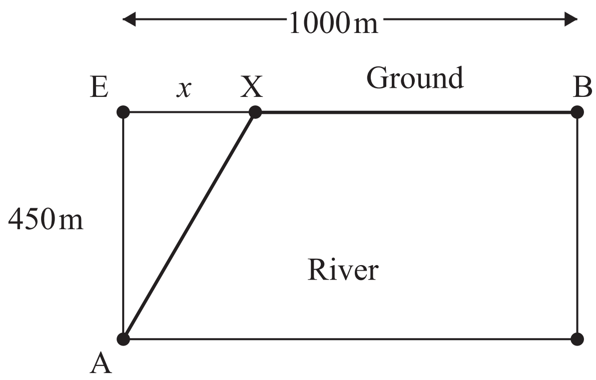
(d) Find $\mathrm{P}\left(X < \dfrac{\pi}{8} \,|\, X < \dfrac{\pi}{4}\right)$. *[2]*

*Do **NOT** write solutions on this page.*

12. *[Maximum mark: 15]*

Engineers need to lay pipes to connect two cities A and B that are separated by a river of width 450 metres as shown in the following diagram. They plan to lay the pipes under the river from A to X and then under the ground from X to B. The cost of laying the pipes under the river is five times the cost of laying the pipes under the ground.

Let $EX = x$.



Let k be the cost, in dollars per metre, of laying the pipes under the ground.

(a) Show that the total cost C, in dollars, of laying the pipes from A to B is given by $C = 5k\sqrt{202500 + x^2} + (1000 - x)k$. *[2]*

(b) (i) Find $\frac{dC}{dx}$.

(ii) Hence find the value of x for which the total cost is a minimum, justifying that this value is a minimum. *[7]*

(c) Find the minimum total cost in terms of k. *[1]*

The angle at which the pipes are joined is $A\hat{X}B = \theta$.

(d) Find θ for the value of x calculated in (b). *[2]*

For safety reasons θ must be at least 120°.

Given this new requirement,

(e) (i) find the new value of x which minimises the total cost;

(ii) find the percentage increase in the minimum total cost. *[3]*

*Do **NOT** write solutions on this page.*

13. *[Maximum mark: 20]*

Consider $z = r(\cos\theta + \mathrm{i}\sin\theta)$, $z \in \mathbb{C}$.

(a) Use mathematical induction to prove that $z^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta)$, $n \in \mathbb{Z}^+$. *[7]*

Given $u = 1 + \sqrt{3}\mathrm{i}$ and $v = 1 - \mathrm{i}$,

(b) (i) express u and v in modulus-argument form;

(ii) hence find u^3v^4. *[4]*

The complex numbers u and v are represented by point A and point B respectively on an Argand diagram.

(c) Plot point A and point B on the Argand diagram. *[1]*

Point A is rotated through $\frac{\pi}{2}$ in the anticlockwise direction about the origin O to become point A'. Point B is rotated through $\frac{\pi}{2}$ in the clockwise direction about O to become point B'.

(d) Find the area of triangle $\mathrm{OA}'\mathrm{B}'$. *[3]*

Given that u and v are roots of the equation $z^4 + bz^3 + cz^2 + dz + e = 0$, where $b, c, d, e \in \mathbb{R}$,

(e) find the values of b, c, d and e. *[5]*

*Do **NOT** write solutions on this page.*

14. *[Maximum mark: 12]*

Particle A moves such that its velocity v ms^{-1}, at time t seconds, is given by $v(t) = \frac{t}{12+t^4}$, $t \geq 0$.

(a) Sketch the graph of $y = v(t)$. Indicate clearly the local maximum and write down its coordinates. *[2]*

(b) Use the substitution $u = t^2$ to find $\int \frac{t}{12+t^4} \mathrm{d}t$. *[4]*

(c) Find the exact distance travelled by particle A between $t = 0$ and $t = 6$ seconds. Give your answer in the form $k \arctan(b)$, $k, b \in \mathbb{R}$. *[3]*

Particle B moves such that its velocity v ms^{-1} is related to its displacement s m, by the equation $v(s) = \arcsin\left(\sqrt{s}\right)$.

(d) Find the acceleration of particle B when $s = 0.1\,\text{m}$. *[3]*

Please **do not** write on this page.

Answers written on this page
will not be marked.